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Viscosity approximation methods for generalized equilibrium problems and fixed point problems

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Abstract The purpose of this paper is to investigate the problem of finding a common element of the set of solutions of a generalized equilibrium problem (for short, GEP) and the set of fixed points of a nonexpansive mapping in the setting of Hilbert spaces. By using well-known Fan-KKM lemma, we derive the existence and uniqueness of a solution of the auxiliary problem for GEP. On account of this result and Nadler's theorem, we propose an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of GEP and the set of fixed points of a nonexpansive mapping. Furthermore, it is proven that the sequences generated by this iterative scheme converge strongly to a common element of the set of solutions of GEP and the set of fixed points of a nonexpansive mapping.

Keywords Viscosity approximation method · Generalized equilibrium problem · Fixed points · Nonexpansive mappings · Strong convergence

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1 Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\| \cdot \|$, respectively. Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(., .)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v) \right\}, \quad \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v \in V} d(u, v)$, $d(U, v) = \inf_{u \in U} d(u, v)$ and $d(u, v) = \|u - v\|$.

Let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $T : C \to CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \to \mathbb{R}$ an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$. We consider the following generalized equilibrium problem (GEP):

$$\text{(GEP)} \quad \begin{cases} \text{Find } u \in C \quad \text{and} \quad w \in T(u) \text{ such that} \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in C. \end{cases}$$

The set of such solutions $u \in C$ of (GEP) is denoted by $(\text{GEP})_S$.

Special Cases:

(i) Given a mapping $N : H \times H \to H$, let $\Phi(w, u, v) = \langle N(w, u), v - u \rangle$ for all $(w, u, v) \in H \times C \times C$, then (GEP) reduces to the following generalized set-valued strongly nonlinear mixed variational inequality problem:

$$\begin{cases} \text{Find } u \in C \quad \text{and} \quad w \in T(u) \text{ such that} \\ \langle N(w, u), v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in C. \end{cases}$$

It has been considered and studied in Zeng et al. (2005) in the case when $C = H$.

(ii) If $\varphi \equiv 0$ and $\Phi(w, u, v) \equiv F(u, v)$ where $F : C \times C \to \mathbb{R}$, then (GEP) becomes the equilibrium problem (for short, EP), which is to find $u \in C$ such that

$$F(u, v) \geq 0, \quad \forall v \in C.$$

The set of such solutions $u \in C$ of EP is denoted by EP_S.

(iii) Given a mapping $T : C \to H$, let $F(u, v) = \langle Tu, v - u \rangle$ for all $(u, v) \in C \times C$, then EP reduces to the classical variational inequality problem, which is to find $u \in C$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in C.$$

For appropriate and suitable choice of the mapping T, the functions Φ and φ, and the convex subset K, one can obtain a number of the known classes of variational inequalities and variational-like inequalities as special cases from (GEP); See, for example, Blum and Oettli (1994); Zeng et al. (2005) and references therein. It is well known that the numerous problems from physics, optimization and economics can be written either in the form of (GEP) or its special cases. In the recent past, some approximation methods have been proposed to solve EP; See, for example, Combettes and Hirstoaga (2005); Flam and Antipin (1997). Combettes and Hirstoaga (2005) introduced an iterative scheme for finding the best approximation to the initial data when EP_S is nonempty and proved a strong convergence theorem.

A mapping $S : C \to H$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(S)$ the set of fixed points of S. Recall that if $C \subseteq H$ is a bounded, closed and convex set and S is a nonexpansive mapping on C into itself, then $F(S) \neq \emptyset$; For fixed point theorems for nonexpansive mappings in Hilbert spaces, we refer to Takahashi (2000), see also, Zeng and Yao (2006a, b). Also, recall that a mapping $f : H \to H$ is contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in H.$$

In 2000, the viscosity approximation method for selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi (2000). He proved the strong convergence of both implicit and explicit iterative schemes in Hilbert spaces setting. Very recently, Takahashi and Takahashi (2007) introduced an iterative scheme by viscosity approximation method for finding a common element of the set of solutions of EP and the set of fixed points of a nonexpansive mapping defined on a Hilbert space. They proved a strong convergence theorem which is connected with Combettes and Hirstoaga's result (Combettes and Hirstoaga 2005) and Wittmann's result (Wittmann 1992).

In this paper, we investigate the problem of finding a common element of the set of solutions of (GEP) and the set of fixed points of a nonexpansive mapping defined on a Hilbert space. On one hand, by using well-known Fan-KKM lemma, we derive the existence and uniqueness of solutions of the auxiliary problems for (GEP). On the other hand, on account of this result and Nadler's theorem, we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (GEP) and the set of fixed points of a nonexpansive mapping. Furthermore, it is proven that the sequences generated by this iterative scheme converge strongly to a common element of the set of solutions of (GEP) and the set of fixed points of a nonexpansive mapping. Our results are the improvements, extension and development of the corresponding results in Combettes and Hirstoaga (2005); Moudafi (2000); Tada and Takahashi (2007) and Takahashi and Takahashi (2007).

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|\cdot\|$. It is well known that for all $x, y \in H$ and $\lambda \in [0, 1]$ there holds

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x - u\| \leq \|x - v\|, \quad \forall v \in C.$$

The mapping $P_C : x \mapsto u$ is called metric projection of H onto C. It is known that P_C is nonexpansive. Further, for all $x \in H$ and $z \in C$,

$$z = P_C x \quad \Leftrightarrow \quad \langle x - z, z - v \rangle \geq 0, \quad \forall v \in C.$$

Lemma 1 (Takahashi 2000) *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Throughout this paper, we shall use the notations "$\rightharpoonup$" and "$\rightarrow$" for weak convergence and strong convergence, respectively.

Definition 1 A multivalued map $T : C \to CB(H)$ is said to be $\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}(T(u), T(v)) \leq \mu\|u - v\|, \quad \forall u, v \in C,$$

where $\mathcal{H}(.,.)$ is the Hausdorff metric on $CB(H)$.

Lemma 2 [Nadler's theorem (Nadler 1969)] *Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(.,.)$ be a Hausdorff metric on $CB(X)$. If $U, V \in CB(X)$, then for any given $\varepsilon > 0$ and $u \in U$, there exists $v \in V$ such that*

$$\|u - v\| \leq (1 + \varepsilon)\mathcal{H}(U, V).$$

For all subset $B \subseteq H$, we denote by $\text{co}(B)$ the convex hull of B. A multivalued mapping $G : B \to 2^H$ is called a KKM mapping if, for every finite subset $\{v_1, v_2, \ldots, v_n\}$ of B,

$$\text{co}(\{v_1, v_2, \ldots, v_n\}) \subseteq \bigcup_{i=1}^{n} G(v_i).$$

In the next section, we shall use the following result.

Lemma 3 (Fan 1961) *Let B be a nonempty subset of a Hausdorff topological vector space E, and let $G : B \to 2^E$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x \in B} G(x) \neq \emptyset$.*

Lemma 4 (Xu 2002) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Proposition 1 [Takahashi 2000; See also Ansari and Yao 2001, pp. 533] *Let D be a nonempty convex subset of H, and $\phi : D \to \mathbb{R}$ be a lower semicontinuous and convex functional. Then ϕ is weakly lower semicontinuous.*

3 Auxiliary problem and iterative schemes

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $T : C \to CB(H)$ be a multivalued map and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function.

To solve the generalized equilibrium problem, let us assume that the equilibrium-like function $\Phi : H \times C \times C \to \mathbb{R}$ satisfies the following conditions with respect to the multivalued map $T : C \to CB(H)$:

(H1) For each fixed $v \in C$, $(w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C$ to $\mathbb{R}$, that is, for $(w, u) \in H \times C$, whenever $w_n \to w$ and $u_n \to u$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \Phi(w_n, u_n, v) \leq \Phi(w, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;
(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

Given $x \in C$ and $w_x \in T(x)$, we consider the following auxiliary problem for GEP (for short, AP(GEP)) that consists of finding $u \in C$ such that

$$\Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \geq 0, \quad \forall v \in C,$$

where $r > 0$ is a constant.

Theorem 1 *Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)–(H3). Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarily and define a mapping $T_r : C \to C$ as follows:*

$$T_r(x) = \left\{ u \in C : \Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \geq 0,\ \forall v \in C \right\}.$$

Then, there hold the following:

(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive (that is, for any $u, v \in C$, $\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v\rangle$) if*

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $w_i \in T(x_i)$, $i = 1, 2$;
(c) *$F(T_r) = (GEP)_S$;*
(d) *$(GEP)_S$ is closed and convex.*

Proof (a) We claim that T_r is a single-valued map. Indeed, for each $x \in C$, take $w_x \in T(x)$ arbitrarily. Then it is sufficient to show the existence and uniqueness of solutions of AP(GEP).

Existence of Solutions of AP(GEP): For the sake of simplicity, we write AP(GEP) as follows: Find $u \in C$ such that

$$r\,[\Phi(w_x, u, v) + \varphi(v) - \varphi(u)] + \langle u - x, v - u\rangle \geq 0, \quad \forall v \in C.$$

For each $v \in C$, we define

$$G(v) = \{z \in C : r\,[\Phi(w_x, z, v) + \varphi(v) - \varphi(z)] + \langle z - x, v - z\rangle \geq 0\}.$$

Note that, for each $v \in C$, $G(v)$ is nonempty since $v \in G(v)$.

We shall prove that G is a KKM mapping. Suppose that there is a finite subset $\{v_1, v_2, \ldots, v_n\}$ of C and $\alpha_i \geq 0$ for all $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$$\hat{v} = \sum_{i=1}^{n} \alpha_i v_i \notin G(v_i), \quad \forall i.$$

Then, we have

$$r\left[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i) - \varphi(\hat{v})\right] + \langle \hat{v} - x, v_i - \hat{v}\rangle < 0, \quad \forall i.$$

Therefore,

$$\sum_{i=1}^{n} \alpha_i r\left[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i) - \varphi(\hat{v})\right] + \sum_{i=1}^{n} \alpha_i \langle \hat{v} - x, v_i - \hat{v}\rangle < 0.$$

Since Φ is an equilibrium-like function, it is easy to see that $\Phi(w_x, \hat{v}, \hat{v}) = 0$. Note that condition (H3) implies the convexity of the functional $v \mapsto \Phi(w_x, \hat{v}, v)$. By using the convexity of φ, we get

$$\begin{aligned} 0 &= r\left[\Phi(w_x, \hat{v}, \hat{v}) + \varphi(\hat{v}) - \varphi(\hat{v})\right] + \langle \hat{v} - x, \hat{v} - \hat{v}\rangle \\ &\leq \sum_{i=1}^{n} \alpha_i r\left[\Phi(w_x, \hat{v}, v_i) + \varphi(v_i) - \varphi(\hat{v})\right] + \sum_{i=1}^{n} \alpha_i \langle \hat{v} - x, v_i - \hat{v}\rangle \\ &< 0, \end{aligned}$$

a contradiction. Hence, G is a KKM mapping.

Since $\overline{G(v)}^w$ [the weak closure of $G(v)$] is a weakly closed subset of a bounded set C in H, it is weakly compact. Hence, by Lemma 3, $\bigcap_{v\in C} \overline{G(v)}^w \neq \emptyset$. Let $u \in \bigcap_{v\in C} \overline{G(v)}^w$. Then, for each $v \in C$, there exists a sequence $\{z_m\}$ in $G(v)$ such that $z_m \rightharpoonup u$. Therefore,

$$r\left[\Phi(w_x, z_m, v) + \varphi(v) - \varphi(z_m)\right] + \langle z_m - x, v - z_m\rangle \geq 0. \tag{1}$$

Since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\begin{aligned} \limsup_{m\to\infty}\langle z_m - x, v - z_m\rangle &= \limsup_{m\to\infty}\left[\langle z_m - x, v\rangle + \langle x, z_m\rangle - \|z_m\|^2\right] \\ &\leq \lim_{m\to\infty}\langle z_m - x, v\rangle + \lim_{m\to\infty}\langle x, z_m\rangle \\ &\quad - \liminf_{m\to\infty}\|z_m\|^2 \\ &\leq \langle u - x, v - u\rangle. \end{aligned}$$

Since φ is a lower semicontinuous and convex functional, it is weakly lower semicontinuous. Note that conditions (H1) and (H2) imply the weakly upper semicontinuity of the functional $z \mapsto \Phi(w_x, z, v)$. Thus, it follows from (1) that

$$\begin{aligned} & r\left[\Phi(w_x, u, v) + \varphi(v) - \varphi(u)\right] + \langle u - x, v - u\rangle \\ &\quad \geq r\left[\limsup_{m\to\infty}\Phi(w_x, z_m, v) + \varphi(v) - \liminf_{m\to\infty}\varphi(z_m)\right] \\ &\qquad + \limsup_{m\to\infty}\langle z_m - x, v - z_m\rangle \\ &\quad \geq \limsup_{m\to\infty}\left\{r\left[\Phi(w_x, z_m, v) + \varphi(v) - \varphi(z_m)\right] + \langle z_m - x, v - z_m\rangle\right\} \\ &\quad \geq 0. \end{aligned}$$

This shows that the auxiliary problem AP(GEP) has a solution $u \in C$.

Uniqueness of Solutions of AP(GEP): Let u_1 and u_2 be two solutions of AP(GEP). Then, for all $v \in C$,

$$r\left[\Phi(w_x, u_1, v) + \varphi(v) - \varphi(u_1)\right] + \langle u_1 - x, v - u_1\rangle \geq 0, \tag{2}$$

and

$$r\left[\Phi(w_x, u_2, v) + \varphi(v) - \varphi(u_2)\right] + \langle u_2 - x, v - u_2\rangle \geq 0. \tag{3}$$

Taking $v = u_2$ in (2) and $v = u_1$ in (3), and adding up these two inequalities, we obtain

$$\begin{aligned} & r\left[\Phi(w_x, u_1, u_2) + \varphi(u_2) - \varphi(u_1)\right] + \langle u_1 - x, u_2 - u_1\rangle \\ &\quad + r\left[\Phi(w_x, u_2, u_1) + \varphi(u_1) - \varphi(u_2)\right] + \langle u_2 - x, u_1 - u_2\rangle \geq 0. \end{aligned} \tag{4}$$

Since Φ is an equilibrium-like function, we have

$$\Phi(w_x, u_1, u_2) + \Phi(w_x, u_2, u_1) = 0,$$

and hence from (4) we get

$$\|u_2 - u_1\|^2 = \langle u_2 - u_1, u_2 - u_1\rangle \le 0.$$

This implies that $u_1 = u_2$. Thus, the solution of AP(GEP) is unique. Therefore, T_r is a single-valued map.
(b) We claim that T_r is firmly nonexpansive. Indeed, for each $x_1, x_2 \in C$, let us define

$$u_1 = T_r(x_1) \quad \text{and} \quad u_2 = T_r(x_2).$$

Then according to the definition of T_r, there are $w_{x_i} \in T(x_i)$ for $i = 1, 2$ such that

$$r\left[\Phi(w_{x_1}, u_1, v) + \varphi(v) - \varphi(u_1)\right] + \langle u_1 - x_1, v - u_1\rangle \ge 0, \tag{5}$$

and

$$r\left[\Phi(w_{x_2}, u_2, v) + \varphi(v) - \varphi(u_2)\right] + \langle u_2 - x_2, v - u_2\rangle \ge 0. \tag{6}$$

Taking $v = u_2$ in (5) and $v = u_1$ in (6), and adding up these two inequalities, we obtain

$$\begin{aligned}&\left[\Phi(w_{x_1}, u_1, u_2) + \varphi(u_2) - \varphi(u_1)\right] + \langle u_1 - x_1, u_2 - u_1\rangle \\ &\quad + r\left[\Phi(w_{x_2}, u_2, u_1) + \varphi(u_1) - \varphi(u_2)\right] + \langle u_2 - x_2, u_1 - u_2\rangle \ge 0. \end{aligned}\tag{7}$$

Since

$$\Phi(w_{x_1}, T_r(x_1), T_r(x_2)) + \Phi(w_{x_2}, T_r(x_2), T_r(x_1)) \le 0,$$

it follows from (7) that

$$0 \le \langle u_1 - x_1, u_2 - u_1\rangle + \langle u_2 - x_2, u_1 - u_2\rangle = \langle x_2 - x_1, u_2 - u_1\rangle - \|u_2 - u_1\|^2,$$

that is,

$$\|T_r(x_2) - T_r(x_1)\|^2 \le \langle T_r(x_2) - T_r(x_1), x_2 - x_1\rangle.$$

This shows that T_r is firmly nonexpansive.
(c) We claim that $F(T_r) = (\text{GEP})_S$. Indeed, we observe that

$$\begin{aligned} x \in F(T_r) &\Leftrightarrow \quad T_r x = x \\ &\Leftrightarrow \quad r\left[\Phi(w_x, x, v) + \varphi(v) - \varphi(x)\right] + \langle x - x, v - x\rangle, \quad \forall v \in C \\ &\Leftrightarrow \quad \Phi(w_x, x, v) + \varphi(v) - \varphi(x) \ge 0, \ \forall v \in C \\ &\Leftrightarrow \quad x \in (\text{GEP})_S. \end{aligned}$$

(d) We claim that $(\text{GEP})_S$ is closed and convex. Indeed, since by conclusion (c), $F(T_r) = (\text{GEP})_S$, it is sufficient to show that $F(T_r)$ is closed and convex. Let $\{x_n\}$ be a sequence in $F(T_r)$ satisfying $x_n \to x \in C$ as $n \to \infty$. Then we have

$$T_r(x) = \lim_{n\to\infty} T_r(x_n) = \lim_{n\to\infty} x_n = x.$$

This implies that $x \in F(T_r)$. Thus $F(T_r)$ is closed.

Now, it remains to show the convexity of $F(T_r)$. Let x_1 and x_2 be any elements of $F(T_r)$. Then for any $t \in [0, 1]$, we write $z = tx_1 + (1-t)x_2$. Observe that

$$\begin{aligned} \|T_r z - z\|^2 &= \|t(T_r z - x_1) + (1-t)(T_r z - x_2)\|^2 \\ &= t\|T_r z - x_1\|^2 + (1-t)\|T_r z - x_2\|^2 - t(1-t)\|x_1 - x_2\|^2 \\ &\le t(1-t)^2\|x_1 - x_2\|^2 + (1-t)t^2\|x_1 - x_2\|^2 \\ &\quad - t(1-t)\|x_1 - x_2\|^2 \\ &= 0, \end{aligned}$$

which implies that $T_r z = z$, that is, $z \in F(T_r)$. Thus, $F(T_r)$ is convex. This completes the proof. □

On account of the above result and Nadler's theorem, we propose an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (GEP) and the set of fixed points of the nonexpansive mapping $S : C \to C$.

Let $\{\alpha_n\}$ be a sequence in [0, 1] and $\{r_n\}$ be a sequence in $(0, \infty)$. Let f be a contraction mapping of C into itself with constant $\alpha \in (0, 1)$. For given $x_1 \in C$ and $w_1 \in T(x_1)$, from Theorem 1, we know that the following auxiliary problem has a unique solution $u_1 = T_{r_1}(x_1) \in C$, that is,

$$\Phi(w_1, u_1, v) + \varphi(v) - \varphi(u_1) + \frac{1}{r_1}\langle u_1 - x_1, v - u_1\rangle \geq 0, \quad \forall v \in C.$$

Utilizing $u_1 \in C$, we define

$$x_2 = \alpha_1 f(x_1) + (1 - \alpha_1)Su_1.$$

Since $w_1 \in T(x_1)$, by Nadler's theorem (Nadler 1969), there exists $w_2 \in T(x_2)$ such that

$$\|w_1 - w_2\| \leq (1 + 1)\mathcal{H}(T(x_1), T(x_2)).$$

By Theorem 1 again, the following auxiliary problem has a unique solution $u_2 = T_{r_2}(x_2) \in C$, that is,

$$\Phi(w_2, u_2, v) + \varphi(v) - \varphi(u_2) + \frac{1}{r_2}\langle u_2 - x_2, v - u_2\rangle \geq 0, \quad \forall v \in C.$$

Utilizing $u_2 \in C$, we define

$$x_3 = \alpha_2 f(x_2) + (1 - \alpha_2)Su_2.$$

Since $w_2 \in T(x_2)$, by Nadler's theorem (Nadler 1969), there exists $w_3 \in T(x_3)$ such that

$$\|w_2 - w_3\| \leq \left(1 + \frac{1}{2}\right)\mathcal{H}(T(x_2), T(x_3)).$$

By induction, we obtain the following iterative algorithm for finding a common element of the set of solutions of (GEP) and the set of fixed points of the nonexpansive mapping $S : C \to C$.

Algorithm 1 For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exist sequences $\{w_n\} \subseteq H$ and $\{x_n\}, \{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n = 1, 2, \ldots. \end{cases} \tag{8}$$

If $S \equiv I$ the identity mapping and $r_n \equiv r > 0$, then Algorithm 1 reduces the following algorithm.

Algorithm 2 For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exist sequences $\{w_n\} \subseteq H$ and $\{x_n\}$, $\{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}\left(T(x_n), T(x_{n+1})\right), \\ \Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n, \quad n = 1, 2, \ldots. \end{cases}$$

4 Strong convergence results

Now, we prove the strong convergence of sequences generated by the Algorithms 1 and 2.

Theorem 2 *Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)–(H3) and S be a nonexpansive mapping of C into itself such that $F(S) \bigcap (GEP)_S \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}$, $\{w_n\}$ and $\{u_n\}$ be sequences generated by (8), where $\{\alpha_n\} \subseteq [0, 1]$ and $\{r_n\} \subseteq (0, \infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If there exists a constant $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2 \tag{9}$$

for all $(r_1, r_2) \in \Xi \times \Xi$, $(x_1, x_2) \in C \times C$ and $w_i \in T(x_i)$, $i = 1, 2$, where $\Xi = \{r_n : n \geq 1\}$, then for $\hat{x} = P_{F(S)\cap(GEP)_S} f(\hat{x})$, there exists $\hat{w} \in T(\hat{x})$ such that $(\hat{x}, \hat{w})$ is a solution of (GEP) and

$$x_n \to \hat{x}, \quad w_n \to \hat{w} \quad \text{and} \quad u_n \to \hat{x}, \quad \textit{as } n \to \infty.$$

Proof It follows from condition (9) that for each $r \in \Xi = \{r_n : n \geq 1\}$

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq -\lambda \|T_r(x_1) - T_r(x_2)\|^2 \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and $w_i \in T(x_i)$, $i = 1, 2$. Hence all conclusions (a)–(d) of Theorem 1 hold. Let $Q = P_{F(S)\cap(\mathrm{GEP})_S}$. Then Qf is a contraction on C into itself.

Indeed, since f is a contraction with constant $\alpha \in (0, 1)$, we have $\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. Therefore, Qf is a contraction of C into itself, which implies that there exists a unique element $q \in C$ such that $q = Qf(q)$.

We divide the remainder of the proof into three steps.

Step 1. We claim that there exist $\hat{x} \in C$ and $\hat{w} \in T(\hat{x})$ such that

$$x_n \to \hat{x}, \quad w_n \to \hat{w} \quad \text{and} \quad u_n \to \hat{x}, \quad \text{as } n \to \infty.$$

Indeed, since C is bounded, the sequences $\{x_n\}$ and $\{u_n\}$ are bounded. Hence, $\{Su_n\}$ and $\{f(x_n)\}$ are also bounded. Next we show that $\|x_{n+1} - x_n\| \to 0$. Observe that

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad = \|\alpha_n f(x_n) + (1-\alpha_n)Su_n - \alpha_{n-1} f(x_{n-1}) - (1-\alpha_{n-1})Su_{n-1}\| \\ &\quad = \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) \\ &\qquad + (1-\alpha_n)Su_n - (1-\alpha_n)Su_{n-1} + (1-\alpha_n)Su_{n-1} \\ &\qquad - (1-\alpha_{n-1})Su_{n-1}\| \\ &\quad \le \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|u_n - u_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|K \\ &\quad \le \alpha_n\alpha\|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|u_n - u_{n-1}\|, \end{aligned} \tag{10}$$

where $K = \sup\{\|f(x_n)\| + \|Su_n\| : n \ge 1\}$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$\Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \ge 0, \quad \forall v \in C, \tag{11}$$

and

$$\Phi(w_{n+1}, u_{n+1}, v) + \varphi(v) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_{n+1} - x_{n+1}, v - u_{n+1}\rangle \ge 0, \tag{12}$$

for all $v \in C$. Putting $v = u_{n+1}$ in (11) and $v = u_n$ in (12), we have

$$\Phi(w_n, u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, u_{n+1} - u_n\rangle \ge 0,$$

and

$$\Phi(w_{n+1}, u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_{n+1} - x_{n+1}, u_n - u_{n+1}\rangle \ge 0.$$

Adding up the last two inequalities, we derive from condition (9)

$$-\lambda\|u_n - u_{n+1}\|^2 + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \ge 0,$$

and hence

$$-r_n\lambda\|u_n - u_{n+1}\|^2 + \left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \ge 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number $b > 0$ such that $r_n > b$ for all $n \ge 1$. Then we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\le -r_n\lambda\|u_{n+1} - u_n\|^2 + \Big\langle u_{n+1} - u_n, x_{n+1} - x_n \\ &\qquad + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\Big\rangle \\ &\le -r_n\lambda\|u_{n+1} - u_n\|^2 + \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| \\ &\qquad + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}, \end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\le -r_n\lambda\|u_{n+1}-u_n\| + \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\le -b\lambda\|u_{n+1}-u_n\| + \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L,\end{aligned}$$

where $L = \sup\{\|u_n - x_n\| : n \ge 1\}$. This implies that

$$\|u_{n+1}-u_n\| \le \theta\|x_{n+1}-x_n\| + (L/b)|r_{n+1}-r_n|, \tag{13}$$

where $\theta = 1/(1+b\lambda)$. Thus, from (10) we have

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\quad\le \alpha_n\alpha\|x_n-x_{n-1}\| + 2|\alpha_n-\alpha_{n-1}|K \\ &\qquad + (1-\alpha_n)(\theta\|x_n-x_{n-1}\| + \frac{L}{b}|r_n-r_{n-1}|) \\ &\quad= (\alpha_n\alpha + (1-\alpha_n)\theta)\|x_n-x_{n-1}\| + 2|\alpha_n-\alpha_{n-1}|K \\ &\qquad + (1-\alpha_n)\frac{L}{b}|r_n-r_{n-1}| \\ &\quad\le \kappa\|x_n-x_{n-1}\| + 2K|\alpha_n-\alpha_{n-1}| + \frac{L}{b}|r_n-r_{n-1}|,\end{aligned} \tag{14}$$

where $\kappa = \max\{\alpha, \theta\} = \max\{\alpha, 1/(1+b\lambda)\}$.

Now put

$$a_n = \|x_n - x_{n-1}\|,\ \gamma_n = 1-\kappa,\ \text{ and }\ \delta_n = 2K|\alpha_n-\alpha_{n-1}| + \frac{L}{b}|r_n-r_{n-1}|,$$

for each $n \ge 1$. Then $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$. Moreover, (14) can be rewritten as

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad \forall n \ge 1.$$

From Lemma 4, we have $a_n \to 0$ as $n \to \infty$, that is, $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Consequently, for any given $\varepsilon > 0$, there exists an integer $N_0 \ge 1$ such that for all integers m and n with $m > n \ge N_0$,

$$\|x_n - x_{n-1}\| < \varepsilon \quad \text{and} \quad \sum_{j=n}^{m}\left[2K|\alpha_j-\alpha_{j-1}| + \frac{L}{b}|r_j-r_{j-1}|\right] < \varepsilon.$$

This together with (14) implies that for all integers m and n with $m > n \ge N_0$,

$$\begin{aligned}\|x_m-x_n\| &\le \sum_{j=n}^{m-1}\|x_{j+1}-x_j\| \\ &\le \kappa\sum_{j=n}^{m-1}\|x_j-x_{j-1}\| + \sum_{j=n}^{m-1}\left[2K|\alpha_j-\alpha_{j-1}| + \frac{L}{b}|r_j-r_{j-1}|\right] \\ &= \kappa\sum_{j=n}^{m-1}\|x_{j+1}-x_j\| - \kappa\|x_m-x_{m-1}\| + \kappa\|x_n-x_{n-1}\| \\ &\quad + \sum_{j=n}^{m-1}\left[2K|\alpha_j-\alpha_{j-1}| + \frac{L}{b}|r_j-r_{j-1}|\right]\end{aligned}$$

$$\leq \kappa \sum_{j=n}^{m-1} \|x_{j+1} - x_j\| + (1+\kappa)\varepsilon,$$

and so

$$\|x_m - x_n\| \leq \sum_{j=n}^{m-1} \|x_{j+1} - x_j\| \leq \frac{1+\kappa}{1-\kappa}\varepsilon. \tag{15}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = \hat{x}$. Since $T : C \to CB(H)$ is a $\mathcal{H}$-Lipschitz continuous multivalued map with constant μ, from (8) we obtain

$$\|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})) \leq \left(1 + \frac{1}{n}\right)\mu\|x_n - x_{n+1}\|.$$

From (15), we conclude that for all integers m and n with $m > n \geq N_0$,

$$\|w_m - w_n\| \leq \sum_{j=n}^{m-1} \|w_{j+1} - w_j\| \leq \sum_{j=n}^{m-1}\left(1 + \frac{1}{j}\right)\mu\|x_{j+1} - x_j\| \leq 2\mu\frac{1+\kappa}{1-\kappa}\varepsilon.$$

Therefore, $\{w_n\}$ is a Cauchy sequence in a complete space H and hence there exists $\hat{w} \in H$ such that $w_n \to \hat{w}$ as $n \to \infty$.

On the other hand, we prove that $\hat{w} \in T(\hat{x})$. Indeed, since $w_n \in T(x_n)$ and

$$\begin{aligned} d(w_n, T(\hat{x})) &\leq \max\left\{d(w_n, T(\hat{x})), \sup_{w\in T(\hat{x})} d(T(x_n), w)\right\} \\ &\leq \max\left\{\sup_{z\in T(x_n)} d(z, T(\hat{x})), \sup_{w\in T(\hat{x})} d(T(x_n), w)\right\} \\ &= \mathcal{H}\left(T(x_n), T(\hat{x})\right). \end{aligned}$$

We derive

$$\begin{aligned} d(\hat{w}, T(\hat{x})) &\leq \|\hat{w} - w_n\| + d(w_n, T(\hat{x})) \\ &\leq \|\hat{w} - w_n\| + \mathcal{H}\left(T(x_n), T(\hat{x})\right) \\ &\leq \|\hat{w} - w_n\| + \mu\|x_n - \hat{x}\| \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $d(\hat{w}, T(\hat{x})) = 0$. Since $T(\hat{x}) \in CB(H)$, it follows that $\hat{w} \in T(\hat{x})$.

Furthermore, from (13) and (15), we obtain for all integers m and n with $m > n \geq N_0$,

$$\begin{aligned} \|u_m - u_n\| &\leq \sum_{j=n}^{m-1} \|u_{j+1} - u_j\| \\ &\leq \sum_{j=n}^{m-1} (\theta\|x_{j+1} - x_j\| + (L/b)|r_{j+1} - r_j|) \\ &\leq \sum_{j=n}^{m-1} \theta\|x_{j+1} - x_j\| + \sum_{j=n}^{m-1} \big[2K|\alpha_{j+1} - \alpha_j| \\ &\quad +(L/b)|r_{j+1} - r_j|\big] \\ &\leq \theta\frac{1+\kappa}{1-\kappa}\varepsilon + \varepsilon \end{aligned}$$

$$= \left(\theta \frac{1+\kappa}{1-\kappa} + 1\right) \varepsilon.$$

Thus $\{u_n\}$ is a Cauchy sequence in C. Let $u_n \to \hat{u} \in C$ as $n \to \infty$.

Since $x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1}) S u_{n-1}$, we have

$$\begin{aligned}\|x_n - Su_n\| &\le \|x_n - Su_{n-1}\| + \|Su_{n-1} - Su_n\| \\ &\le \alpha_{n-1}\|f(x_{n-1}) - Su_{n-1}\| + \|u_{n-1} - u_n\|.\end{aligned}$$

Since $\alpha_n \to 0$, we have $\|x_n - Su_n\| \to 0$. For $p \in F(S) \bigcap (\mathrm{GEP})_S$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right),\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|Su_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|x_n - u_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\quad -\|x_{n+1} - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 \\ &\quad +\|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

So, we have $\|x_n - u_n\| \to 0$. Since

$$\|u_n - \hat{x}\| \le \|u_n - x_n\| + \|x_n - \hat{x}\|,$$

we have $u_n \to \hat{x}$.

Step 2. We claim that $\hat{x} \in F(S) \cap (\mathrm{GEP})_S$. Indeed, since

$$\begin{aligned}\|S\hat{x} - \hat{x}\| &\le \|S\hat{x} - Su_n\| + \|Su_n - x_n\| + \|x_n - u_n\| + \|u_n - \hat{x}\| \\ &\le \|Su_n - x_n\| + \|x_n - u_n\| + 2\|u_n - \hat{x}\|,\end{aligned}$$

we have $S\hat{x} = \hat{x}$. Hence, $\hat{x} \in F(S)$. Let us show $\hat{x} \in (\mathrm{GEP})_S$. By $u_n = T_{r_n} x_n$, we have

$$\Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \ge 0, \quad \forall v \in C.$$

Since $\lim_{n\to\infty}\|u_n - x_n\| = 0$, from condition (H1) and the lower semicontinuity of φ we derive

$$\Phi(\hat{w}, \hat{x}, v) + \varphi(v) - \varphi(\hat{x}) \ge 0, \quad \forall v \in C.$$

This implies that $\hat{x} \in (\mathrm{GEP})_S$. Consequently, $\hat{x} \in F(S) \cap (\mathrm{GEP})_S$.

Step 3. We claim that $\hat{x} = q$, where $q = P_{F(S)\cap(\mathrm{GEP})_S} f(q)$. Indeed, by Step 2 we have

$$\lim_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \langle f(q) - q, \hat{x} - q\rangle \le 0. \tag{16}$$

Note that $x_{n+1} - q = (1-\alpha_n)(Su_n - q) + \alpha_n(f(x_n) - q)$. Then by Lemma 1, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1-\alpha_n)^2\|Su_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&= (1-\alpha_n)^2\|Su_n - q\|^2 \\
&\quad +2\alpha_n\langle f(x_n) - f(q) + f(q) - q, x_{n+1} - q\rangle \\
&\le (1-\alpha_n)^2\|u_n - q\|^2 + 2\alpha_n\alpha\|x_n - q\|\|x_{n+1} - q\| \\
&\quad +2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\
&\le (1-\alpha_n)^2\|u_n - q\|^2 + \alpha_n\alpha\{\|x_n - q\|^2 + \|x_{n+1} - q\|^2\} \\
&\quad +2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - q\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(q) - q, x_{n+1} - q\rangle \\
&= \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\right)\|x_n - q\|^2 + \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n - q\|^2\right. \\
&\quad \left. + \frac{1}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle\right\}.
\end{aligned}$$

Put $\gamma_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}$. Then $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\lim_{n\to\infty}\gamma_n = 0$. Since

$$\lim_{n\to\infty}\left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n - q\|^2 + \frac{1}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle\right\} = 0,$$

by Lemma 4, we know that $\lim_{n\to\infty}\|x_n - q\| = 0$. According to the uniqueness of the limit, we get $q = \hat{x}$. This completes the proof. □

Example 1 We now illustrate condition (9) by virtue of an example.

Let $H = C = \mathbb{R}$ and let the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ in $H = \mathbb{R}$ be defined in the usual sense as

$$\langle x, y\rangle = xy,\ \forall x, y \in H, \quad \text{and} \quad \|x\| = |x|,\ \forall x \in H.$$

Let r be a positive constant with $0 < r < \frac{1}{2}$ and put $r_n = r,\ \forall n \ge 1$. Then $\Xi := \{r_n : n \ge 1\} = \{r\}$.

Let us define $T : C \to CB(H)$ and $\Phi : H \times C \times C \to \mathbb{R}$ as follows:

$$Tx = 2x,\ \forall x \in H, \quad \text{and} \quad \Phi(w, x, y) = \langle w, y - x\rangle,\ \forall (w, x, y) \in H \times C \times C.$$

Let x_1 and x_2 be arbitrary elements in C. Observe that for $w_1 = Tx_1 = 2x_1$ and $w_2 = Tx_2 = 2x_2$,

$$\begin{aligned}
&\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1) \\
&\quad = \langle w_1, T_r x_2 - T_r x_1\rangle + \langle w_2, T_r x_1 - T_r x_2\rangle \\
&\quad = \langle 2x_1, T_r x_2 - T_r x_1\rangle + \langle 2x_2, T_r x_1 - T_r x_2\rangle \\
&\quad = -2\langle x_1 - x_2, T_r x_1 - T_r x_2\rangle.
\end{aligned} \tag{17}$$

On the other hand, let us define

$$u_1 = T_r x_1 \quad \text{and} \quad u_2 = T_r x_2.$$

Then according to the definition of T_r, there hold the following inequalities for $w_1 = T x_1 = 2x_1$ and $w_2 = T x_2 = 2x_2$,

$$r[\Phi(w_1, u_1, v) + \varphi(v) - \varphi(u_1)] + \langle u_1 - x_1, v - u_1\rangle \geq 0, \tag{18}$$

and

$$r[\Phi(w_2, u_2, v) + \varphi(v) - \varphi(u_2)] + \langle u_2 - x_2, v - u_2\rangle \geq 0. \tag{19}$$

Taking $v = u_2$ in (18) and $v = u_1$ in (19), and adding up these two inequalities, we obtain

$$\begin{aligned} 0 &\leq r[\Phi(w_1, u_1, u_2) + \varphi(u_2) - \varphi(u_1)] + \langle u_1 - x_1, u_2 - u_1\rangle \\ &\quad + r[\Phi(w_2, u_2, u_1) + \varphi(u_1) - \varphi(u_2)] + \langle u_2 - x_2, u_1 - u_2\rangle \\ &= r[\Phi(w_1, u_1, u_2) + \Phi(w_2, u_2, u_1)] + \langle u_1 - x_1, u_2 - u_1\rangle \\ &\quad + \langle u_2 - x_2, u_1 - u_2\rangle \\ &= r[\Phi(w_1, u_1, u_2) + \Phi(w_2, u_2, u_1)] + \langle x_2 - x_1, u_2 - u_1\rangle - \|u_2 - u_1\|^2 \\ &= r[\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1)] + \langle x_2 - x_1, T_r x_2 - T_r x_1\rangle \\ &\quad - \|T_r x_2 - T_r x_1\|^2, \end{aligned}$$

which together with (17), implies that

$$\begin{aligned} \|T_r x_2 - T_r x_1\|^2 &\leq r[\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1)] \\ &\quad + \langle x_2 - x_1, T_r x_2 - T_r x_1\rangle \\ &= r[\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1)] \\ &\quad - \frac{1}{2}[\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1)] \\ &= -\left(\frac{1}{2} - r\right)[\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1)]. \end{aligned}$$

Therefore, it follows that

$$\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1) \leq -\lambda \|T_r x_2 - T_r x_1\|^2,$$

where $\lambda = 2/(1-2r)$. This shows that inequality (9) holds for all $(r_1, r_2) \in \Xi \times \Xi$, $(x_1, x_2) \in C \times C$ and $w_i \in T(x_i),\ i = 1, 2$.

As a direct consequence of Theorem 2, we obtain the following result.

Corollary 1 *Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be a $\mathcal{H}$-Lipschitz continuous multivalued map with constant μ, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)–(H3) and S be a nonexpansive mapping of C into itself such that $F(S) \cap (\mathrm{GEP})_S \neq \emptyset$. Let r be a positive parameter, f be a contraction of C into itself and $\{x_n\}$, $\{w_n\}$ and $\{u_n\}$ be sequences generated by Algorithm 2, where $\{\alpha_n\} \subseteq [0, 1]$ satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

If there exists a constant $\lambda > 0$ *such that*

$$\Phi(w_1, T_r x_1, T_r x_2) + \Phi(w_2, T_r x_2, T_r x_1) \leq -\lambda \|T_r x_1 - T_r x_2\|^2$$

for all $(x_1, x_2) \in C \times C$ *and* $w_i \in T(x_i)$, $i = 1, 2$, *then for* $\hat{x} = P_{F(S) \cap (\mathrm{GEP})_S} f(\hat{x})$, *there exists* $\hat{w} \in T(\hat{x})$ *such that* $(\hat{x}, \hat{w})$ *is a solution of (GEP) and*

$$x_n \to \hat{x}, \quad w_n \to \hat{w} \quad \text{and} \quad u_n \to \hat{x}, \quad \text{as } n \to \infty.$$

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